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Purity-bounded uncertainty relations in multidimensional space—generalized purity

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Abstract

Uncertainty relations for mixed quantum states (precisely, purity-bounded position–momentum relations, developed by Bastiaans and then by Man'ko and Dodonov) are studied in the general multidimensional case. An expression for a family of mixed states at the lower bound of uncertainty relation is obtained. It is shown that in the case of entropy-bounded uncertainty relations, the lower-bound state is thermal, and a transition from the one-dimensional problem to the multidimensional one is trivial. Results of numerical calculation of the relation of the lower bound for different types of generalized purity are presented. Analytical expressions for general purity-bounded relations for highly mixed states are obtained.

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1. Introduction and review

A well-known position–momentum uncertainty relation for standard deviations of $\hat{x}$ and $\hat{p}$ operators,

$$\Delta x \Delta p \geqslant \hbar/2, \tag{1}$$

is valid for any state (described either by a wavefunction or by a density matrix [1, 2]) and represents a class of inequalities which play quite an important role in quantum physics. In particular, uncertainty relations [3, 4] set the precision limits of measurement process for non-commuting observables. Another important example is that generalized coherent states (and squeezed states) could be defined as a set of states which minimize an uncertainty relation (see [5]). The uncertainty principle and properties of its minimum are also of special interest in the theory of operators in Hilbert space [6].

Inequality (1) has been generalized to include extra dependence on degree of purity [7] of a quantum state

$$\mu = \mathrm{Tr}(\hat{\rho}^2) \tag{2}$$

($\hat{\rho}$ is a density operator), the parameter $0 \leqslant \mu \leqslant 1$ and equality $\mu = 1$ is achieved only for pure states. An asymptotic inequality for one-dimensional highly mixed states with $\mu \ll 1$ has a form [8–14]

$$\Delta x \Delta p \geqslant \frac{8}{9\mu} \frac{\hbar}{2}. \tag{3}$$

In addition to the trace (2) of the squared density operator, there are other measures of overall purity (see the above-cited papers for details, especially a recent comprehensive review on purity-bounded relations [14]).

Another approach for treatment of the uncertainty relation for mixed states was developed by Wolf, Ponomarenko and Agarwal [15, 16], and also by Vourdas and his co-authors [17, 18]. In the cited works, the uncertainty relation is expressed in terms of correlations of respective observables. On the other hand, the inequality of type (3) relates uncertainties in conjugated variables and a measure of the overall purity of the state.

A generalization of the uncertainty relation (1) to multidimensional space (vector observables, which can appear e.g. for multimode states, or multi-particle situations) was investigated in the early days of quantum mechanics [19] (see also a review in [13]) and is still drawing the attention of researches [20–22]. In its most simple form, the uncertainty relation for n-dimensional position and momentum operators $\hat{X} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$, $\hat{P} = (\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n)$ could be written as

$$(\Delta X \Delta P)^n \geqslant \left(\frac{\hbar}{2}\right)^n, \tag{4}$$

with definitions

$$(\Delta X)^2 = \frac{1}{n} \prod_{i=1}^{n} (\Delta x_i)^2, \qquad (\Delta P)^2 = \frac{1}{n} \prod_{i=1}^{n} (\Delta p_i)^2.$$

In fact, due to equality between different coordinates in the minimum of the uncertainty relation, inequality (4) has the same meaning as the nth degree of the standard one-dimensional relation (1), see also discussion in [23].

The problem of generalization of inequality (3) for the multidimensional case was treated in papers by Karelin and Lazaruk [23, 24]. In particular, in our first paper on this topic [23], it was shown with the help of the Wigner function formalism that there is a non-trivial dependence of the purity-bounded uncertainty relation limit on the number of dimensions. For highly mixed states with $\mu \ll 1$,

$$(\Delta X \Delta P)^n \geqslant \frac{C(n)}{\mu} \left(\frac{\hbar}{2}\right)^n, \qquad C(n) = \frac{2^{n+1}(n+1)!}{(n+2)^{n+1}}, \tag{5}$$

where the parameter $C(n)$ characterizes the distance from a minimum $(\hbar/2)^n$ for pure states. In deriving (5), we have assumed that the Wigner function of the minimum-uncertainty state is nonnegative.

In another paper [24], the structure of the density matrix near the lower bound of uncertainty relation was also found, using decomposition of the density matrix in terms of Fock states (which form an orthogonal basis with a minimal uncertainty)

$$\hat{\rho} = \sum_{\mathbf{m},\mathbf{m}'} a_{\mathbf{m},\mathbf{m}'} |m_1\rangle |m_2\rangle \cdots |m_n\rangle \langle m_1'| \langle m_2'| \cdots \langle m_n'|, \tag{6}$$

with the additional condition

$$\mu = \sum_{\mathbf{m},\mathbf{m}'} |a_{\mathbf{m},\mathbf{m}'}|^2 = \text{const},$$

where $\mathbf{m} = (m_1, \ldots, m_s)$ is a 'vector' index with integer nonnegative components.

At the lower bound of the uncertainty relation, the density matrix in the Fock representation is diagonal, $a_{\mathbf{m},\mathbf{m}'} = a_{\mathbf{m},\mathbf{m}}\delta_{\mathbf{m},\mathbf{m}'}$; coefficients $a_{\mathbf{m},\mathbf{m}}$ depend linearly on the 'norm' of vector index $\|\mathbf{m}\| = \sum_{i=1}^{n} m_i$, and the quantity of Fock states in the representation of $\hat{\rho}$ is finite. Coefficients $a_{\mathbf{m},\mathbf{m}}$ of this decomposition are degenerate, and their multiplicity is determined by norm $\|\mathbf{m}\|$ and dimensionality n:

$$g_{\|\mathbf{m}\|}^{(n)} = \frac{(\|\mathbf{m}\| + n - 1)!}{(\|\mathbf{m}\|)!(n-1)!}. \tag{7}$$

The inequality obtained in [24] correctly describes the whole range of μ, including the perfectly pure case $\mu = 1$. In particular, in the interpolating form it becomes

$$\Delta X \Delta P \geqslant \frac{\hbar}{2}\frac{n + 2L(\mu)}{n+2} \tag{8}$$

with the auxiliary real parameter $L(\mu)$ being a root of the transcendental equation

$$\mu = \frac{(n+2L)(n+1)!\Gamma(L)}{(n+2)\Gamma(L+n+1)}, \tag{9}$$

where $\Gamma(y)$ is Euler's gamma-function.

It is also necessary to note that the inequality, mathematically practically the same as the uncertainty relation, but with another physical meaning, is often used for classical wave fields, e.g. in optics [8–12, 23]. The results of the present paper, as well as of preceding papers [23, 24] could be used, with appropriate change of notations, for classical partially coherent fields and sources (in one-, two- and three-dimensional space [25]).

The uncertainty relations (5) and (8) could be further generalized in order to take into account the dependence of the inequality minimum on the eigenvalues of the density operator. A preliminary report on this topic, with the stress on partially coherent classical fields, was published in [26]. Obtaining such a relation, together with the study of its asymptotics, is the main aim of the present paper.

2. Uncertainty relation for the diagonal representation of the density matrix

Any density matrix $\hat{\rho}$ has a spectral decomposition [27]

$$\hat{\rho} = \sum_m \rho_m |\psi_m\rangle\langle\psi_m|, \tag{10}$$

where ρ_m are the eigenvalues, and $|\psi_m\rangle$ are eigenvectors of the density operator; then, each of vectors $|\psi_m\rangle$ could be represented via outer products of one-dimensional Fock states $|k\rangle$

$$|\psi_m\rangle = \sum_{k_1,k_2,\ldots,k_n} A_{k_1,k_2,\ldots,k_n}^{(m)} |k_1\rangle|k_2\rangle\cdots|k_n\rangle, \tag{11}$$

where k_i, $i = 1, 2, \ldots, n$, corresponds to the ith one-dimensional subspace.

The right-hand side of the uncertainty relation (4) is calculated using the method of papers [8, 13]. The core idea of the calculation is the introduction of the auxiliary observable

$$E(\vartheta) = \tfrac{1}{2}[(\Delta P)^2/\vartheta + \vartheta(\Delta X)^2], \tag{12}$$

which could be regarded as the energy of some oscillator with unit frequency and mass ϑ. The minimum of $E(\vartheta)$ with respect to ϑ (for $\vartheta = \Delta X/\Delta P$) is exactly the left-hand side of the uncertainty relation

$$\min_{\vartheta} E(\vartheta) = \Delta X \Delta P.$$

In the one-dimensional case, the Fock states $|k\rangle$ are eigenstates of the harmonic oscillator with eigenvalues $2k+1$; then substitution of (10) and (11) into (12) leads to

$$\Delta X \Delta P \geqslant \frac{\hbar}{2}\frac{1}{n}\sum_m \rho_m \sum_{k_1,k_2,\ldots,k_n} [2(k_1+k_2+\cdots+k_n)+n]\left|A^{(m)}_{k_1,k_2,\ldots,k_n}\right|^2. \tag{13}$$

Now, due to isomorphism between the set of all positive integers and the set of combinations of n positive integers, it is possible to consider the coefficients $A^{(m)}_{k_1,k_2,\ldots,k_n}$ as elements of some unitary matrix $\{\tilde{A}_{\mathbf{k}m}\}$. Then, (in)equality (13) can be treated with a lemma[1] from [10] to give

$$\Delta X \Delta P \geqslant \frac{\hbar}{2}\frac{1}{n}\sum_{k_1,k_2,\ldots,k_n} [2(k_1+k_2+\cdots+k_n)+n]\rho_{m(k_1,k_2,\ldots,k_n)}, \tag{14}$$

where eigenvalues of the density matrix are ordered in a non-increasing sequence.

Dependence of the expression $2(k_1+k_2+\cdots+k_n)+n$ on indices $k_1, k_2, \ldots, k_n$ is degenerate: this expression takes the same values for several combinations of indices. Therefore, it is possible to rewrite the (in)equality (14) as

$$\Delta X \Delta P \geqslant \frac{\hbar}{2}\frac{1}{n}\sum_k (2k+n)\sum_{m=0}^{g_m^{(n)}-1} \rho_{m(k)} \tag{15}$$

where the values $g_m^{(n)}$ (degeneration multiplicity) are defined by formula (7), and the eigenvalues of the density matrix are collected in groups of $g_m^{(n)}$ terms. Expression (15) is the main result of the paper, and it is the most general form of the uncertainty relation for mixed states (partially coherent fields) in a multidimensional space. This inequality relates a minimal uncertainty volume of a state to the spectrum of the density operator corresponding to this state.

3. Multidimensional purity-bounded relations

Using the method from papers [10, 11], it is possible to find a dependence of the uncertainty relation limit on some characteristics of purity of a quantum system. Usually, a family of 'generalized purities' (Shatten p-norms or 'generalized entropies' [22]) is used, which is defined as

$$\mu^{(r)} = [\mathrm{Tr}(\hat{\rho}^{r/(r-1)})]^{r-1}, \tag{16}$$

where r is an arbitrary (not necessary integer) real number with $r > 1$. Important special cases of $\mu^{(r)}$ include $\mu^{(2)} = \mu$ ('usual' purity, see above), 'superpurity'

$$\mu^{(1)} = \lim_{r\to 1} \mu^{(r)}, \tag{17}$$

when only the largest eigenvalue of the density matrix is taken into account, and also 'entropy-based' purity degree

$$\mu_S = \exp(-S), \qquad S = -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho}), \tag{18}$$

which is defined in terms of Shannon–von Neumann entropy S, and so leads to the 'entropy-bounded' uncertainty relation. As is shown in [12], μ_S can be treated as a limiting case of definition (16) for $r \to \infty : \mu^{(\infty)} = \mu_S$.

[1] See also [14]. To be self-contained, the lemma is reproduced, together with necessary changes for multidimensional case, in appendix A of this paper.

'Superpurity' and entropy-bounded uncertainty relations play a special role for the one-dimensional case: owing to continuous non-increasing dependence of $\mu^{(r)}$ on r for $r \geqslant 1$ [12], they are limiting cases of the family of characteristics (16). It is also possible to show that the non-increasing dependence of $\mu^{(r)}$ on r remains valid in a general multidimensional case (see details in appendix B).

As can easily be shown by the Lagrange method, at the minimum of uncertainty relation, the definition (16) reduces to

$$\mu = \left[\sum_k g_k^{(n)} \left(\xi_k / g_k^{(n)}\right)^{r/(r-1)}\right]^{r-1}, \tag{19}$$

where

$$\xi_k = \sum_{m=0}^{g_s^{(n)}-1} \rho_{m(k)}$$

and

$$\sum_k \xi_k = 1. \tag{20}$$

Then the (in)equality (15) may be rewritten as

$$\Delta X \Delta P \geqslant \frac{\hbar}{2} \frac{1}{n} \sum_k (2k+n)\xi_k \tag{21}$$

and, in order to obtain the relation of type (5) for given structure of the density matrix, (i.e. eigenvalues ρ_m) it is necessary to find a minimum with respect to variables ξ_s.

The task of detailed study of the uncertainty relation minimum has, in general, no analytical solution (the same as in the one-dimensional case [14]). Besides interpolated and asymptotic inequalities, which will be studied later in the paper, it is possible to obtain an analytical solution for the case of entropy-bounded relations.

Using the Lagrange method, it is easy to show that a minimum of uncertainty product $\Delta X \Delta P$ (21) for given entropy S is attained if the coefficients ξ_m are given by

$$\xi_m = A g_m^{(n)} \exp(-\beta m), \tag{22}$$

where A is a normalization constant and parameter β depends on the entropy. Taking into account the structure of the density operator at the minimum of the general uncertainty relation (15) (compare representation (11)), in the case of the entropy-bounded relation the density matrix takes the form

$$\hat{\rho}_S^{(n)} = \underbrace{\hat{\rho}_S^{(1)} \hat{\rho}_S^{(1)} \ldots \hat{\rho}_S^{(1)}}_{n \text{ times}}. \tag{23}$$

Here

$$\hat{\rho}_S^{(1)} = (1 - \mathrm{e}^{-\beta}) \sum_{k=0}^{\infty} \mathrm{e}^{-\beta k} |k\rangle\langle k| \tag{24}$$

is a density matrix corresponding to the minimum of the one-dimensional entropy-bounded relation. Parameter β can be found from the solution of the transcendental equation

$$\frac{S}{n} = \frac{\beta}{\exp(\beta) - 1} - \ln(1 - \exp(-\beta)), \tag{25}$$

which is in accordance with the appropriate equation for one-dimensional case [13].

The uncertainty relation then could be written as

$$(\Delta X \Delta P)^n \geqslant \left(\frac{2}{\exp(\beta)-1}+1\right)^n \left(\frac{\hbar}{2}\right)^n, \tag{26}$$

or, for highly-mixed states with $S \gg 1$ (and $\beta \ll 1$ with $\beta \approx \exp(1-S/n)$)

$$(\Delta X \Delta P)^n \geqslant \exp(S)\left(\frac{2}{\mathrm{e}}\right)^n \left(\frac{\hbar}{2}\right)^n. \tag{27}$$

The obtained structure of eigenstate decomposition is factorized on solutions of the one-dimensional problem (see [11, 13]), which leads to the thermal state (24). In other words, the entropy-bounded uncertainty relation of position–momentum type has no additional effects for multidimensional cases.

In order to study the general low-purity case, it is possible to utilize the approach from Bastiaans' paper [10], which is based on generalization of the Hölder inequality. The mathematical details are presented in appendix C. Defining an 'uncertainty function' $C(\mu^{(r)}, n)$ similarly to (5), it follows that

$$C(\mu^{(r)}, n) = \mu^{(r)}\left(\frac{1}{n}\{2M+n-2[\mu^{(r)}B(M,n,r)]^{1/r}\}\right)^n, \tag{28}$$

where

$$B(M,n,r) = \sum_{0\leqslant m\leqslant M} \frac{(m+n-1)!}{(n-1)!m!}(M-n)^r \tag{29}$$

and an additional real and positive minimization parameter M is introduced.

The 1D problem has the known asymptotical solution [10]

$$C(\mu^{(r)}, 1) = 2[r/(r+1)]^r, \qquad \mu^{(r)} \ll 1.$$

In the same way, highly mixed states could be treated analytically for the arbitrary multidimensional case: as far as the limit of small $\mu^{(r)}$ requires M to be sufficiently large [24, 14], it is possible to replace the summation in formula (29) by an integration. Together with the approximation of degeneration multiplicity by $m^{n-1}/(n-1)!$ it gives

$$B(M,n,r) \approx \frac{1}{(n-1)!}\int_0^M m^{n-1}(M-m)^r\,\mathrm{d}m. \tag{30}$$

The last relation could be calculated analytically,

$$B(M,n,r) \approx M^{n+r+1}\left(\prod_{k=1}^{n+1}(r+k)\right)^{-1} \tag{31}$$

(see appendix D for details). Further minimization of relation (28) with respect to M, after some tedious but quite elementary algebra, gives an asymptotic variant of the general purity bounded uncertainty relation

$$(\Delta X \Delta P)^n \geqslant \left(\frac{\hbar}{2}\right)^n \frac{C(n,r)}{\mu^{(r)}}, \qquad C(n,r) = \frac{2^n r^r}{(n+r)^{n+r}}\prod_{k=1}^{n}(r+k), \tag{32}$$

which describes the whole range of n and r for $\mu^{(r)} \ll 1$ (see figure 1). Dependence of $C(n,t)$ on r for $n = 2, 3$ together with the one-dimensional case is presented in figure 2. It is seen that the obtained expressions demonstrate a decrease of the uncertainty minimum with increase of r, leading to entropy-bounded relations at $r \to \infty$.

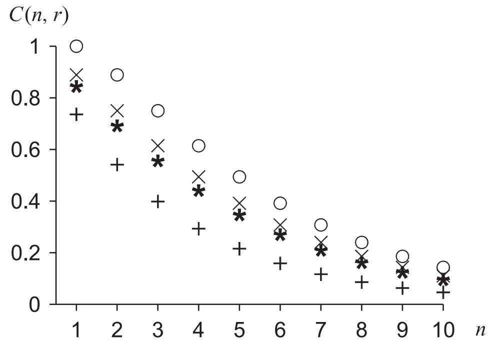

Figure 1. Uncertainty relation minimum for $\mu^{(r)} \ll 1, n = 1, \ldots, 6$ and different variants of the degree of purity: $\circ$—$r \to 1$, $\times$—$r = 2$, $*$—$r = 3$, $+$—$r \to \infty$ (entropy-bounded relation).

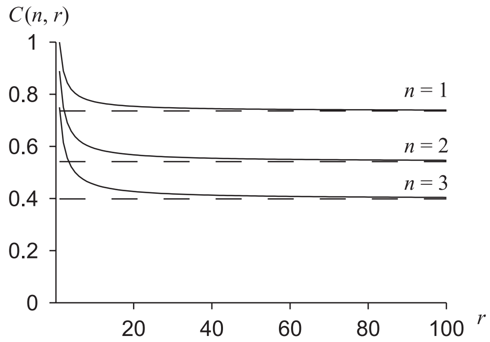

Figure 2. Uncertainty relation minimum for $\mu^{(r)} \ll 1, n = 1, 2, 3$ and different variants of the degree of purity (solid lines, $r = 1, \ldots, 100$). Dashed lines denote the uncertainty minimum for entropy-bounded relation with $r \to \infty$ (for $n = 1, 2, 3$, respectively).

4. Concluding remarks

To summarize, it is worth noting that two main forms of the uncertainty principle (of position-momentum type) for mixed states in multidimensional space are obtained in the present paper. The first one (15) relates the minimal uncertainty product with the eigenspectrum of the density matrix and the second (32) is an asymptotic purity-bounded uncertainty relation for small generalized purity. In both cases, a minimum of the uncertainty product is obtained when the eigenstates of the density operator are Fock states. In the case of the purity-bounded relation, the eigenspectrum of the density operator is defined by

$$\xi_m \propto g_m^{(n)} (M - m)^{r-1}, \qquad 0 \leqslant m \leqslant M \tag{33}$$

(see appendix C). In other words, the spectral representation of the density matrix is a finite sum of Fock states. Such a state is definitely non-classical, see the discussion in Dodonov's paper [14]. Upon transition to $r \to \infty$ (entropy-bounded relations), the minimum-uncertainty state becomes thermal (24), i.e. classical. A more detailed study of minimum-uncertainty state structure will be the subject of another publication.

It is also necessary to note that the results of the paper are applicable for analysis and characterization of entangled quantum states. Indeed, the spectral decomposition of the density matrix is closely connected to the Schmidt decomposition of non-separable states (see, e.g. [28]). The approach to the uncertainty principle for entangled states can be based on the mathematically analogous case of uncertainty (reciprocity) relations for a pulsed partially coherent classical beam [25].

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Appendix A

The lemma is reproduced here mainly for completeness of the material, and in order to make the above analysis clearer. Initially it was presented in appendix A of paper [10]; it also can be found in [14]. According to [10], the idea of this proof was initially proposed by M L J Hautus.

Let the sequence of numbers b_m be defined by

$$b_m = \sum_{k=0}^{\infty} |a_{mk}|^2 \gamma_k, \tag{A.1}$$

where $\gamma_0 \leqslant \gamma_1 \leqslant \cdots \leqslant \gamma_k \leqslant \cdots$ and coefficients a_{mk} satisfy the orthonormality condition

$$\sum_{k=0}^{\infty} a_{mk} a_{lk}^* = \delta_{ml}, \qquad m, l = 0, 1, \ldots. \tag{A.2}$$

One may consider the numbers b_m for $m = 0, 1, \ldots, M$ as the diagonal entries of an $(M+1)$-square Hermitian matrix $H = \|h_{ij}\|$ with

$$h_{ij} = \sum_{k=0}^{\infty} a_{ik} a_{jk}^* \gamma_k, \qquad i, j = 0, 1, \ldots, M. \tag{A.3}$$

Let the eigenvalues β of H be ordered according to

$$\beta_0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_k \leqslant \cdots \leqslant \beta_M. \tag{A.4}$$

From Cauchy's inequalities for eigenvalues of a submatrix of a Hermitian matrix [29], we can conclude that $\beta_m \geqslant \gamma_m$ $(m = 0, 1, \ldots, M)$ and hence

$$\sum_{m=0}^{M} b_m = \sum_{m=0}^{M} h_{mm} = \sum_{m=0}^{M} \beta_m \geqslant \sum_{m=0}^{M} \gamma_m. \tag{A.5}$$

Furthermore, with the numbers λ_m (or ρ_m, in this paper) satisfying the property $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_m \geqslant \cdots$, we can formulate the chain of relations

$$\begin{aligned}
\sum_{m=0}^{M} \lambda_m b_m &= \lambda_0 b_0 + \sum_{m=1}^{M} \lambda_m b_m \\
&= \lambda_0 b_0 + \sum_{m=1}^{M} \lambda_m \left[\sum_{l=0}^{m} b_l - \sum_{l=0}^{m-1} b_l \right] \\
&= \lambda_0 b_0 + \sum_{m=1}^{M} \lambda_m \sum_{l=0}^{m} b_l - \sum_{m=0}^{M-1} \lambda_{m+1} \sum_{l=0}^{m} b_l \\
&= \sum_{m=0}^{M-1} \lambda_m \sum_{l=0}^{m} b_l + \lambda_M \sum_{l=0}^{M} b_l - \sum_{m=0}^{M-1} \lambda_{m+1} \sum_{l=0}^{m} b_l
\end{aligned}$$

$$= \lambda_M \sum_{l=0}^{M} b_l + \sum_{m=0}^{M-1} (\lambda_m - \lambda_{m+1}) \sum_{l=0}^{m} b_l$$
$$\geqslant \lambda_M \sum_{l=0}^{M} \gamma_l + \sum_{m=0}^{M-1} (\lambda_m - \lambda_{m+1}) \sum_{l=0}^{m} \gamma_l$$
$$= \sum_{m=0}^{M} \lambda_m \gamma_m. \tag{A.6}$$

On choosing $\gamma_n = 2n + 1$ and taking the limit $M \to \infty$, we arrive at the inequality

$$\sum_{m=0}^{\infty} \lambda_m \sum_{n=0}^{\infty} |a_{mn}|^2 (2n+1) \geqslant \sum_{m=0}^{\infty} \lambda_m (2m+1),$$

which becomes an equality if $|a_{mn}| = \delta_{mn}$.

In order to modify this proof to miltidimensional case, it is necessary to choose γ_m as $\gamma_{\mathbf{m}} = 2(m_1 + \cdots + m_s) + n$ (here $\mathbf{m} = (m_1, \ldots, m_n)$ is a 'vectorial' summation index), and then to take into account degeneracy of coefficients $\gamma_{\mathbf{m}}$.

Appendix B

By analogy with Bastiaans' paper [12] for any r, q, with $1 < r < q$, it holds that

$$\mu^{(r)} = \left[\sum_m g_m^{(n)} \theta_m^{r/(r-1)}\right]^{r-1} = \left[\sum_m g_m^{(n)} \left(\theta_m^{q/(q-1)}\right)^{(q-1)/(r-1)} (\theta_m)^{(r-q)/(r-1)}\right]^{r-1}$$
$$\leqslant \left[\left(\sum_m g_m^{(n)} \theta_m^{q/(q-1)}\right)^{(q-1)/(r-1)} \left(\sum_m g_m^{(n)} \theta_m\right)^{(r-q)/(r-1)}\right]^{r-1} = \mu^{(q)}. \tag{B.1}$$

Here $\theta_m = \xi_m / g_m^{(n)}$, and the Hölder inequality for the weighted sum [30] is used, see formula (C.3). Therefore, for a family of purities (16), it is possible to conclude that 'superpurity' and entropy-based purity lead to limiting cases of all multidimensional uncertainty relations.

Appendix C

Starting from equation (20), with ξ_m a sequence of nonnegative numbers and M an arbitrary real nonnegative constant, we obtain

$$\sum_{m=0}^{\infty} \xi_m = \frac{1}{2M+n}\left[2\sum_{m=0}^{\infty} \xi_m (M-m) + \sum_{m=0}^{\infty} \xi_m (2m+n)\right] = 1. \tag{C.1}$$

The following (in)equalities hold

$$\sum_{m=0}^{\infty} \xi_m (M-m) \leqslant \sum_{0\leqslant m\leqslant M} \xi_m (M-m), \tag{C.2}$$

$$\sum_{0\leqslant m\leqslant M} \xi_m (M-m) = \sum_{0\leqslant m\leqslant M} g_m^{(n)} (M-m) \xi_m / g_m^{(n)}$$
$$\leqslant \left[\sum_{0\leqslant m\leqslant M} g_m^{(n)} (M-m)^r\right]^{1/r} \left[\sum_{0\leqslant m\leqslant M} g_m^{(n)} \left(\xi_m / g_m^{(n)}\right)^p\right]^{1/p}, \tag{C.3}$$

$$\sum_{0\leqslant m\leqslant M} g_m^{(n)}\left(\xi_m/g_m^{(n)}\right)^p \leqslant \sum_{m=0}^{\infty} g_m^{(n)}\left(\xi_m/g_m^{(n)}\right)^p \tag{C.4}$$

with two real parameters $p, r \geqslant 1$, $1/p + 1/r = 1$. The equality sign in relations (C.2) and (C.4) holds if $\xi_m =$ for $m > M$. Relation (C.3) changes to equality, when $\xi_m \propto g_m^{(n)}(M-m)^{r-1}$ in the interval $0 \leqslant m \leqslant M$. (In)equality (C.3) is a general form of the Hölder inequality for the weighted sum [30], see also [31].

Combining (in)equalities (C.1)–(C.4) gives a relation

$$\frac{1}{2M+n}\left[2B(M,n,r)^{1/r}\mu_q + \sum_{m=0}^{\infty}\xi_m(2m+n)\right] \geqslant 1, \tag{C.5}$$

where

$$B(M,n,r) = \sum_{0\leqslant m\leqslant M} g_m^{(n)}(M-m)^r, \tag{C.6}$$

$$\mu_p = \left[\sum_{m=0}^{\infty} g_m^{(n)}\left(\xi_m/g_m^{(n)}\right)^p\right]^{1/rp}. \tag{C.7}$$

From the condition $1/p + 1/r = 1$ it follows that $p = r/(r-1)$ and then (in)equality (28) results.

Appendix D

In order to find an integral in approximation (30), we start from the introduction of a new variable $x = M - m$

$$B(M,n,r) \approx \frac{1}{(n-1)!}\int_0^M \mathrm{d}x\,(M-x)^{n-1}x^r. \tag{D.1}$$

Application of the binomial formula to $(M-x)^{n-1}$ and interchanging the order of integration and summation leads to

$$B(M,n,r) \approx M^{n+r+1}\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!(k+r+1)}. \tag{D.2}$$

The last sum can be calculated by use of formula (5.41) from the book by Graham, Knuth and Patashnik [32], leading at last to (31).

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